Identification of discrete-time model

Introduction to Process (System) Identification

Process (system) identification is the field of modeling dynamics systems from experimental data. For example, if a dynamics system can be represented by a difference equation

\[ y_n + a_1 y_{n-1} + \ldots + a_m y_{n-m} = b_1 u_{n-1} + \ldots + b_p u_{n-p} \]

then the task of system identification is to estimate the parameters \( \{a_1, a_2, \ldots, a_m, b_1, \ldots, b_p\} \) using plant input and output data \( \{u_0, u_1, \ldots, u_N\} \) and \( \{y_0, y_1, \ldots, y_N\} \).

The schematic diagram of system identification

Types Of Model

Models of dynamics system can be of many kinds, including the following:

- Mental, intuitive or verbal models
- Graphs and tables
- Mathematical models \( \rightarrow \) a description of a system in term of equations

Mathematical Modeling and System Identification

- Mathematical modeling or first principle modeling.
- Process or system identification.

Why does one perform system identification?
Properties of the models obtained from system identification

- limited validity
- little physical insight → the parameters are used only as tools to give a good description of the system's overall behavior
- relatively easy to construct and use

Identification is not a foolproof methodology that can be used without interaction from the user → reasons:

- An appropriate model structure must be found.
- There are certainly no “perfect” data in real life.
- The process may vary with time
- It may be difficult or impossible to measure some variables signals

System Identification Procedure

ARX or Differential Equation Model Identification

The nature of linear difference equation models is such that it is particularly convenient to estimate the unknown parameters using linear regression.

\[ y_n = a_1 y_{n-1} + a_2 y_{n-2} + \cdots + a_m y_{n-m} + \]
\[ + b_1 u_{n-1} + b_2 u_{n-2} + \cdots + b_n u_{n-n} + e_n \]

where \( e_n \) is the equation error. This is also regarded as an ARX model. The transfer function form of this ARX model is

\[ \frac{y_n}{u_n} = \frac{b_1 q^{-1} + b_2 q^{-2} + \cdots + b_m q^{-m}}{1 - a_1 q^{-1} - a_2 q^{-2} - \cdots - a_m q^{-m}} \]
For example, a second order ARX model can be written as

\[ y_k = a_1 y_{k-1} + a_2 y_{k-2} + b_1 u_{k-2} + b_2 u_{k-1} + e_k \]  

Suppose that after having excited the system with the input sequence: \( \{ u_1, u_2, \ldots, u_{N-1}, u_N \} \), the process response data \( \{ y_1, y_2, \ldots, y_{N-1}, y_N \} \) are measured. Starting from \( n = 3 \) in equation (1), we have:

\[
\begin{align*}
    y_1 & = a_1 y_0 + a_2 y_1 + b_1 u_2 + b_2 u_1 + e_1 \\
    y_2 & = a_1 y_1 + a_2 y_2 + b_1 u_3 + b_2 u_2 + e_2 \\
    \vdots & \quad \vdots \\
    y_{N-1} & = a_1 y_{N-2} + a_2 y_{N-3} + b_1 u_{N-1} + b_2 u_{N-2} + e_{N-1} \\
    y_N & = a_1 y_{N-1} + a_2 y_{N-2} + b_1 u_{N-2} + b_2 u_{N-3} + e_N 
\end{align*}
\]

The parameters \( \{ a_1, a_2, b_1, b_2 \} \) are unknown and need at least 4 equations to solve for these four unknown parameters. Because of the equation error \( \{ e_3, e_4, \ldots, e_N, e_{N+1} \} \), the solution \( \{ a_1, a_2, b_1, b_2 \} \) will not be the exactly the same as \( \{ a_1, a_2, b_1, b_2 \} \).

An estimation method to minimize the error is the least squares estimator.

Write the algebraic eq in vector-matrix form:

\[
\begin{bmatrix}
    y_1 \\
    y_2 \\
    \vdots \\
    y_{N-1} \\
    y_N 
\end{bmatrix} =
\begin{bmatrix}
    y_2 & y_3 & u_2 & u_3 \\
    y_3 & y_4 & u_3 & u_4 \\
    \vdots & \vdots & \vdots & \vdots \\
    y_{N-2} & y_{N-3} & u_{N-2} & u_{N-3} \\
    y_{N-1} & y_{N-2} & u_{N-1} & u_{N-2} 
\end{bmatrix}
\begin{bmatrix}
    a_1 \\
    a_2 \\
    b_1 \\
    b_2 
\end{bmatrix}
\]

which is of the form: \( Y = X \theta \)

The least squares estimate is then given by

\[
\hat{\theta} = (X^T X)^{-1} X^T Y
\]

where \( \hat{\theta} \) is the estimate of \( \theta \) and given by

\[
\begin{bmatrix}
    a_1 \\
    a_2 \\
    b_1 \\
    b_2 
\end{bmatrix}
\]
Notice that the difference equation (1) has a time delay one unit. This can be seen by writing the transfer function form of equation (1) using backshift operator as

\[
y[n] = \frac{b_0 q^{-1} + b_0 q^{-2}}{1 - a_1 q^{-1} - a_0 q^{-2}}
\]  

(2)

Suppose now the process has more than one unit time delay, say, three unit delays. Then the second order difference equation with three unit time delays can be written as

\[
y[n] = a_1 y[n-1] + a_2 y[n-2] + b_1 u[n-3] + b_2 u[n-4] + e[n]
\]

Suppose that after having excited the system with the input sequence \(\{u_1, u_2, \ldots, u_{N-1}, u_N\}\), the process response data \(\{y_1, y_2, \ldots, y_{N-1}, y_N\}\), are measured. Starting from \(n = 5\) in equation (2), we have:

\[
\begin{align*}
y_5 &= a_1 y_4 + a_2 y_3 + b_1 u_2 + b_2 u_1 + e_3 \\
y_6 &= a_1 y_5 + a_2 y_4 + b_1 u_3 + b_2 u_2 + e_4 \\
&\quad \vdots \\
y_{N-1} &= a_1 y_{N-2} + a_2 y_{N-3} + b_1 u_{N-4} + b_2 u_{N-5} + e_{N-1} \\
y_N &= a_1 y_{N-1} + a_2 y_{N-2} + b_1 u_{N-3} + b_2 u_{N-4} + e_N
\end{align*}
\]

Write the algebraic equations in vector-matrix form, we obtain

\[
\begin{bmatrix}
y_5 \\
y_6 \\
\vdots \\
y_{N-1} \\
y_N
\end{bmatrix} =
\begin{bmatrix}
y_4 \\
y_3 \\
\vdots \\
y_{N-2} \\
y_{N-1}
\end{bmatrix}
\begin{bmatrix}
a_1 \\
a_2 \\
b_1 \\
b_2
\end{bmatrix}
\begin{bmatrix}
u_2 \\
u_1 \\
\vdots \\
u_{N-4} \\
u_{N-3}
\end{bmatrix} +
\begin{bmatrix}
e_3 \\
e_4 \\
\vdots \\
e_{N-1} \\
e_N
\end{bmatrix}
\]

which is of the form: \(Y = X\theta\)

The least squares estimate is then given by

\[
\hat{\theta} = (X^T X)^{-1} X^T Y
\]